# The Hachemeister Regression Model

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*In this article we will obtain, just like in the case of classical credibility model, a credibility solution in the form of a linear combination of the individual estimate (based on the data of a particular state) and the collective estimate (based on aggregate USA data).*
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## Introduction

The article contains a description of the Hachemeister regression model allowing for effects like inflation.

In **Section 1** we give Hachemeister's original model, which involves only one isolated contract. His original model, involving only one contract, contains the basics of all further regression credibility models. In this section we will give the assumption of the Hachemeister regression credibility models and the optimal linearized regression credibility premium is derived.

**Section 2** describes the classical Hachemeister model. In the classical Hachemeister model, a portfolio of contracts is studied. Just as in **Section 1** we will derive the best linearized regression credibility premium for this model and we will provide some useful estimators for the structure parameters.

## 1. The original regression credibility model of Hachemeister

In the original regression credibility model of Hachemeister, we consider one contract with unknown and fixed risk parameter $\theta$, during a period of t ($\geq 2$) years. The yearly claim amounts are denoted by $X_1,...,X_t$. Suppose $X_1,...,X_t$ are random variables with finite variance. The contract is a random vector consisting of a random structure parameter $\theta$ and observations $X_1,...,X_t$. Therefore, the contract is equal to $(\theta, \underline{X}')$, where $\underline{X}' = (X_1,...,X_t)$. For this model we want to estimate the net premium: $\mu(\theta) = E(X_j \mid \theta),\ j = \overline{1,t}$ for a contract with risk parameter $\theta$.

**Remark**

In the credibility models, the pure net risk premium of the contract with risk parameter $\theta$ is defined as:

$$\mu(\theta) = E(X_j \mid \theta), \forall j = \overline{1,t} \qquad (1.1)$$

Instead of assuming time independence in the net risk premium (1.1) one could assume that the conditional expectation of the claims on a contract changes in time, as follows:

$$\mu_j(\theta) = E(X_j \mid \theta) = \underset{\sim}{Y}'_j \underset{\sim}{b}(\theta), \forall j = \overline{1,t} \quad (1.2)$$

where the design vector $\underset{\sim}{Y}_j$ is known ($\underset{\sim}{Y}_j$ is a column vector of length q, the non-random (q×1) vector $\underset{\sim}{Y}_j$ is known) and where the $\underset{\sim}{b}(\theta)$ are the unknown regression constants ($\underset{\sim}{b}(\theta)$ is a column vector of length q).

**Remark**

Because of inflation we are not willing to assume that $E(X_j \mid \theta)$ is independent of j. Instead we make the regression assumption $E(X_j \mid \theta) = \underset{\sim}{Y}'_j \underset{\sim}{b}(\theta)$.

When estimating the vector $\underset{\sim}{\beta}$ from the initial regression hypothesis $E(X_j) = \underset{\sim}{Y}'_j \underset{\sim}{\beta}$ formulated by actuary, Hachemeister found great differences. He then assumed that to each of the states there was related an unknown random risk parameter $\theta$ containing the risk characteristics of the state, and that's from different states were independent and identically distributed. Again considering one particular state, we assume that $E(X_j \mid \theta) = \underset{\sim}{Y}'_j \underset{\sim}{b}(\theta)$, with $E[\underset{\sim}{b}(\theta)] = \underset{\sim}{\beta}$.

After these motivating introductory remarks, we state the model assumptions in more detail.

Let $\underset{\sim}{X} = (X_1, ..., X_t)'$ be an observed random $(t \times 1)$ vector and $\theta$ an unknown random risk parameter. We assume that:

$$(\mathbf{H_1})\ E(\underset{\sim}{X} \mid \theta) = \underset{\sim}{Y}\,\underset{\sim}{b}(\theta).$$

It is assumed that the matrices:

$$(\mathbf{H_2})\ \underset{\sim}{\Lambda} = Cov[\underset{\sim}{b}(\theta)],\ \left(\underset{\sim}{\Lambda} = \underset{\sim}{\Lambda}^{(q\times q)}\right)$$

$$(\mathbf{H_3})\ \underset{\sim}{\Phi} = E[Cov(\underset{\sim}{X} \mid \theta)],\ \left(\underset{\sim}{\Phi} = \underset{\sim}{\Phi}^{(t\times t)}\right)$$

are positive definite. We finally introduce:

$$(\mathbf{H_4})\ E[\underset{\sim}{b}(\theta)] = \underset{\sim}{\beta}$$

Let $\tilde{\mu}_j$ be the credibility estimator of $\mu_j(\theta)$ based on $\underset{\sim}{X}$. The optimal choice of $\tilde{\mu}_j$ is determined in the following theorem:

*Theorem 1.1*: The credibility estimator $\tilde{\mu}_j$ is given by:

$$\tilde{\mu}_j = \underset{\sim}{Y}'_j \left[ \underset{\sim}{Z}\,\hat{\underset{\sim}{b}} + \left(I - \underset{\sim}{Z}\right)\underset{\sim}{\beta} \right] \quad (1.3),$$

$$\text{with: } \hat{\underset{\sim}{b}} = \left(\underset{\sim}{Y}\,\underset{\sim}{\Phi}^{-1}\,\underset{\sim}{Y}\right)^{-1} \underset{\sim}{Y}'\,\underset{\sim}{\Phi}^{-1}\,\underset{\sim}{X} \quad (1.4),$$

$$\underset{\sim}{Z} = \underset{\sim}{\Lambda}\,\underset{\sim}{Y}'\,\underset{\sim}{\Phi}^{-1}\,\underset{\sim}{Y}\left(\underset{\sim}{I} + \underset{\sim}{\Lambda}\,\underset{\sim}{Y}'\,\underset{\sim}{\Phi}^{-1}\,\underset{\sim}{Y}\right)^{-1} \quad (1.5),$$

where $\underset{\sim}{I}$ denotes the $q \times q$ identity matrix $\left(\hat{\underset{\sim}{b}} = \underset{\sim}{\hat{b}}^{(q\times 1)};\ \underset{\sim}{z} = \underset{\sim}{z}^{(q\times q)}\right)$ for some fixed j. For demonstration see **[1]** from References.

**Remark**

By the credibility estimator of a vector we shall mean the vector of the credibility estimator of each element of the vector to be estimated. Let $\tilde{\mu}$ be the credibility estimator of the vector $\underset{\sim}{\mu}(\theta) = \underset{\sim}{Y}\,\underset{\sim}{b}(\theta)$, where the non-random $(t \times q)$ matrix $\underset{\sim}{Y}$ is known. Then we clearly have:

$$\underset{\sim}{\tilde{\mu}} = \underset{\sim}{Y}\left[ \underset{\sim}{Z}\,\hat{\underset{\sim}{b}} + \left(\underset{\sim}{I} - \underset{\sim}{Z}\right)\underset{\sim}{\beta} \right] \quad (1.6)$$

An interesting special case is $\underset{\sim}{Y} = \underset{\sim}{I}$, where we get:

$$\underset{\sim}{\tilde{b}} = \underset{\sim}{Z}\,\hat{\underset{\sim}{b}} + \left(\underset{\sim}{I} - \underset{\sim}{Z}\right)\underset{\sim}{\beta} \quad (1.7)$$

as the credibility estimator of $\underset{\sim}{b}(\theta)$.

We introduce the following definition:

*Definition:*

Let $r(\theta)$ be a real-valued function of $\theta$ and $\hat{r}$ an estimator of $r(\theta)$. We shall say that $\hat{r}$ is a $\theta$-unbiased estimator of $r(\theta)$, if $E\left(\hat{r} \mid \theta\right) = r(\theta)$ a.s..

By the requirement of $\theta$-unbiasedness for $\mu_j(\theta)$, we obtain the following theorem:

*Theorem 1.2:* The best linear $\theta$-unbiased estimator of $\mu_j(\theta)$ based on $\underset{\sim}{X}$ is:

$$\tilde{\mu}_j = \underset{\sim}{Y}'_j\,\hat{\underset{\sim}{b}} \quad (1.8)$$

for some fixed $j = \overline{1,t}$. For demonstration see [1] from References.

**Remark**

As by the credibility estimators, we directly transfer the results of *Theorem 1.2*, to estimation of vectors and get that $\underset{\sim}{\hat{\mu}} = \underset{\sim}{Y}\,\hat{\underset{\sim}{b}}$ is the best linear $\theta$-unbiased estimator of $\underset{\sim}{\mu}(\theta) = \underset{\sim}{Y}\,\underset{\sim}{b}(\theta)$ and as a special case, that $\hat{\underset{\sim}{b}}$ is the best linear $\theta$-unbiased estimator of $\underset{\sim}{b}(\theta)$.

This last result gives the following interpretation of (1.7).

*Theorem 1.3:* The credibility estimator of $\underset{\sim}{b}(\theta)$ (so $\underset{\sim}{\tilde{b}}$) is a weighted mean of the best linear $\theta$-unbiased estimator of $\underset{\sim}{b}(\theta)$ (so $\hat{\underset{\sim}{b}}$) and the expectation of $\underset{\sim}{b}(\theta)$ (so $\underset{\sim}{\beta}$).

## 2. The classical credibility regression model of Hachemeister

In this section we will introduce the classical regression credibility model of Hachemeister, which consists of a portfolio of k contracts, satisfying the constraints of the original Hachemeister model.

The contract indexed j is a random vector consisting of a random structure $\theta_j$ and observations $X_{j1},...,X_{jt}$. Therefore the contract indexed j is equal to $\left(\theta_j, \underline{X}_j^{'}\right)$, where $\underline{X}_j^{'} = \left(X_{j1},...,X_{jt}\right)$ and $j=\overline{1,k}$ (the variables describing the $j^{th}$ contract are $\left(\theta_j, \underline{X}_j^{'}\right), j=\overline{1,k}$). Just as in **Section 1** we will derive the best linearized regression credibility estimators for this model.

Instead of assuming time independence in the net risk premium:

$$\mu(\theta_j) = E(X_{jq} \mid \theta_j), j=\overline{1,k}, q=\overline{1,t} \quad (2.1)$$

one could assume that the conditional expectation of the claims on a contract changes in time, as follows:

$$\mu_q(\theta_j) = E(X_{jq} \mid \theta_j) = y_{jq}\beta(\theta_j), j=\overline{1,k}, q=\overline{1,t} \quad (2.2),$$

with $y_{jq}$ assumed to be known and $\beta(\cdot)$ assumed to be unknown.

*Consequence of the hypothesis (2.2)*:

$$\underline{\mu}^{(t,1)}(\theta_j) = E(\underline{X}_j \mid \theta_j) = x^{(t,n)}\underline{\beta}^{(n,1)}(\theta_j), j=\overline{1,k} \quad (2.3),$$

where $x^{(t,n)}$ is a matrix given in advance, the so-called design matrix, and where the $\underline{\beta}(\theta_j)$ are the unknown regression constants. Again one assumes that for each contract the risk parameters $\underline{\beta}(\theta_j)$ are the same functions of different realizations of the structure parameter.

For some fixed design matrix $x^{(t,n)}$ of full rank n ($n<t$), and a fixed weight matrix $v_j^{(t,t)}$, the hypotheses of the Hachemeister model are:

(H1) The contracts $\left(\theta_j, \underline{X}_j^{'}\right)$ are independent; the variables $\theta_1,...,\theta_k$ are independent and identically distributed.

(H2) $E\left(\underline{X}_j^{(t,1)} \mid \theta_j\right) = x^{(t,n)}\underline{\beta}^{(n,1)}(\theta_j), j=\overline{1,k}$, where $\underline{\beta}$ is an unknown regression vector;

$Cov\left(\underline{X}_j^{(t,1)} \mid \theta_j\right) = \sigma^2(\theta_j)v_j^{(t,t)}$, where: $\sigma^2(\theta_j) = Var(X_{jr} \mid \theta_j), \forall r=\overline{1,t}$ and $v_j = v_j^{(t,t)}$ is a known non-random weight $(t \times t)$ matrix, with $\operatorname{rg} v_j = t, j=\overline{1,k}$.

We introduce the structural parameters, which are natural extensions of those in the Bühlmann-Straub model. We have:

$$s^2 = E\left[\sigma^2(\theta_j)\right] \quad (2.4)$$

$$a = a^{(n,n)} = Cov\left[\underline{\beta}(\theta_j)\right] \quad (2.5)$$

$$\underline{b} = \underline{b}^{(n,1)} = E\left[\underline{\beta}(\theta_j)\right] \quad (2.6),$$

where $j=\overline{1,k}$.

After the credibility result based on these structural parameters is obtained, one has to construct estimates for these parameters. Write:

$$c_j = c_j^{(t,t)} = Cov(\underline{X}_j)$$

$$u_j = u_j^{(n,n)} = \left(x' v_j^{-1} x\right)^{-1}$$

$z_j = z_j^{(n,n)} = a\left(a + s^2 u_j\right)^{-1} =$ [the resulting credibility factor for contract j], $j=\overline{1,k}$.

We can now derive the regression credibility results for the estimates of the parameters in the linear model. Multiplying this vector of the estimates by the design matrix provides us with the credibility estimate for $\underline{\mu}(\theta_j)$, see (2.3).

*Theorem 2.1*: (Linearized regression credibility premium)

The best linearized estimate of $E\left[\underline{\beta}^{(n,1)}(\theta_j) \mid \underline{X}_j\right]$ is given by:

$$\underline{M}_j = z_j^{(n,n)}\underline{B}_j^{(n,n)} + \left(I - z_j^{(n,n)}\right)\underline{b}^{(n,1)} \quad (2.7)$$

and the best linearized estimate of $E\left[x^{(t,n)}\underline{\beta}^{(n,1)}(\theta_j) \mid \underline{X}_j\right]$ is given by:

$$x^{(t,n)}\underline{M}_j = x^{(t,n)}\left[z_j^{(n,n)}\underline{B}_j^{(n,n)} + \left(I - z_j^{(n,n)}\right)\underline{b}^{(n,1)}\right] \quad (2.8),$$

where $\underline{B}_j$ is the classical result for the regression vector, namely the GLS–estimator for $\underline{\beta}(\theta_j)$ [the vector $\underline{B}_j$ minimizing the weighted distance to the observations $\underline{X}_j, d(\underline{B}_j) = (\underline{X}_j - x\underline{B}_j)' v_j^{-1} (\underline{X}_j - x\underline{B}_j)$, reads

$$\underline{B}_j = (x' v_j^{-1} x)^{-1} x' v_j^{-1} \underline{X}_j = u_j x' v_j^{-1} \underline{X}_j = (x' c_j^{-1} x)^{-1} x' c_j^{-1} \underline{X}_j$$

, in case

$c_j = s^2 v_j + xax'$ ], $j = \overline{1,k}$.

For proof see [1] from References.

**Remark**

From (2.7) we see that the credibility estimates for the parameters of the linear model are given as the matrix version of a convex mixture of the classical regression result $\underline{B}_j$ and the collective result $\underline{b}$.

*Theorem 2.1* concerns a special contract j. By the assumption, the structural parameters $a, \underline{b}$ and $s^2$ do not depend on j. So if there are more contracts, these parameters can be estimated.

Every vector $\underline{B}_j$ gives an unbiased estimator of $\underline{b}$. Consequently, so does every linear combination of type $\sum_j \alpha_j \underline{B}_j$, where the vector of matrices $(\alpha_j^{(n,n)})_{j=\overline{1,k}}$, is such that:

$$\sum_{j=1}^{k} \alpha_j^{(n,n)} = I^{(n,n)} \qquad (2.9)$$

The optimal choice of $\alpha_j^{(n,n)}$ is determined in the following theorem:

*Theorem 2.2*: (Estimation of the parameters $\underline{b}$ in the regression credibility model)

The optimal solution to the problem $\underset{\underline{\alpha}}{Min}\, d(\underline{\alpha})$ (2.10), where:

$$d(\underline{\alpha}) = \left\| \underline{b} - \sum_j \alpha_j \underline{B}_j \right\|_P^2 \overset{def.}{=} E\left[ \left( \underline{b} - \sum_j \alpha_j \underline{B}_j \right)' P \left( \underline{b} - \sum_j \alpha_j \underline{B}_j \right) \right] = \text{(the distance from}$$

$$\left( \sum_j \alpha_j \underline{B}_j \right) \text{ to the parameters } \underline{b}\text{)},$$

$P = P^{(n,n)}$ a given positive definite matrix ($P$ is a non-negative definite matrix), with the vector of matrices $\underline{\alpha} = (\alpha_j)_{j=\overline{1,k}}$ satisfying (2.9), is:

$$\underline{\hat{b}}^{(n,1)} = Z^{-1} \sum_{j=1}^{k} z_j \underline{B}_j \quad (2.11),$$

where $Z = \sum_{j=1}^{k} z_j$ and $z_j$ is defined as:

$$z_j = a(a + s^2 u_j)^{-1}, j = \overline{1,k}$$

For the proof see [1] the chapter 8 from References.

*Theorem 2.3*: (Unbiased estimator for $s^2$ foe each contract group)

In case the number of observations $t_j$ in the $j^{th}$ contract is larger than the number of regression constants n, the following is an unbiased estimator of $s^2$:

$$\hat{s}_j^2 = \frac{1}{t_j - n} (\underline{X}_j - x_j \underline{B}_j)' v_j^{-1} (\underline{X}_j - x_j \underline{B}_j) \quad (2.12)$$

For the proof see [1]., the chapter 8, from References.

*Corollary:* (Unbiased estimator for $s^2$ in the regression model)

Let K denote the number of contracts j, with $t_j > n$. Then $E\left(\hat{s}^2\right) = s^2$, if:

$$\hat{s}^2 = \frac{1}{K} \sum_{j; t_j > n} \hat{s}_j^2 \quad (2.13)$$

For a, we give an unbiased pseudo-estimator, defined in terms of itself, so it can only be computed iteratively:

*Theorem 2.4*: (Pseudo-estimator for a)

The following random variable has expected value a:

$$\hat{a} = \frac{1}{k-1} \sum_j z_j \left( \underline{B}_j - \underline{\hat{b}} \right) \left( \underline{B}_j - \underline{\hat{b}} \right)' \quad (2.14)$$

For the proof see [1]., the chapter 8, from References.

*Remark:*

Another unbiased estimator for a is the following:

$$\hat{a} = 1/\left(w_{.}^2 - \sum w_j^2\right)\left\{\frac{1}{2}\sum_{i,j} w_i w_j \left(\underline{B}_i - \underline{B}_j\right)\left(\underline{B}_i - \underline{B}_j\right)' - \hat{s}^2 \sum_{j=1}^{k} w_j \left(w_{.} - w_j\right)u_j\right\}, \quad (2.15)$$

where $w_j$ is the volume of the risk for the $j^{th}$ contract, $j = \overline{1,k}$ and $w_{.} = \sum_{j=1}^{k} w_j$ .

*Observation:*
This estimator is a statistic; it is not a pseudo-estimator. Still, the reason to prefer (2.14) is that this estimator can easily be generalized to multi-level hierarchical models. In any case, the unbiasedness of the credibility premium disappears even if one takes (2.15) to estimate a.

**Conclusions**
The article contains a credibility solution in the form of a linear combination of the individual estimate (based on the data of a particular state) and the collective estimate (based on aggregate USA data). This idea is worked out in regression credibility theory.
In case there is an increase (for instance by inflation) of the results on a portfolio, the risk premium could be considered to be a linear function in time of the type $\beta_0(\theta)+t\beta_1(\theta)$. Then two parameters $\beta_0(\theta)$ and $\beta_1(\theta)$ must be estimated from the observed variables. This kind of problem is named regression credibility. This model arises in cases where the risk premium depends on time, e.g. by inflation. The one could assume a linear effect on the risk premium as an approximation to the real growth, as is also the case in time series analysis.
These regression models can be generalized to get credibility models for general regression models, where the risk is characterized by outcomes of other related variables.
This paper contains a description of the Hachemeister regression model allowing for effects like inflation. If there is an effect of inflation, it is contained in the claim figures, so one should use estimates based on these figures instead of external data. This can be done using Hachemeister's regression model.
In this article the regression credibility result for the estimate of the parameters in the linear model is derived. After the credibility result based on the structural parameters is obtained, one has to construct estimates for these parameters.
The mathematical theory provides the means to calculate useful estimators for the structure parameters.
The property of unbiasedness of these estimators is very appealing and very attractive from point of view practical.
The fact that it is based on complicated mathematics, involving conditional expectations and conditional covariances, needs not bother the user more than it does when he applies statistical tools like discriminant analysis, scoring models, SAS and GLIM.

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